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THE THEOREM OF ROTATION IN ELEMENTARY MECHANICS.¹

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In order to discuss the motion of a rigid body in a plane, we need to know two things: (1) the translational motion of some one point of the body; and (2) the angular motion of some one line of the body.

Information in regard to the first point is obtained from the well known theorem on the motion of the center of mass.

Information in regard to the second point is obtained from a theorem which is variously known as the theorem of rotation, the theorem of moments, the theorem of angular acceleration, etc.

This theorem of rotation is usually written in the form

$$T = I\dot{\omega},$$

where T = the torque, or turning moment, of the external forces about some suitably chosen point Q , I = the moment of inertia of the body about that point, and $\dot{\omega}$ = the angular acceleration of the body.

In order to bring out the "dimensions" of the quantities involved, the equation may be written in the more explicit form

$$\Sigma Fp = \frac{W}{g} k^2 \frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the given point Q , k = the radius of gyration of the body about that point, and $d\omega/dt$ = the angular acceleration of the body at the instant in question.

This equation may be called the equation obtained by "taking moments about the point Q ," and the question at once arises: for what choices of the point Q will the equation be valid? In other words, about what points is it legitimate to "take moments"?

Surprising as it may seem, the elementary text books are not in agreement in their answer to this question.

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All the books do indeed agree that the equation is true if Q is a fixed point, that is, a point fixed in the body and fixed in space; and also if Q is the center of mass. That is, it is always legitimate to take moments about a fixed axis, or about the center of mass.

But many books go further and state that the equation is true when Q is the instantaneous center, and solve many problems by taking moments about the instantaneous center. For example we mention only the familiar problem of the motion of a cylinder rolling down an inclined plane, which is often solved by taking moments about the point of contact.

And finally, some books go so far as to state that the equation is true no matter what point may be taken as the point Q .¹

In view of these discrepancies among current text books, it seems worth while to review the development of this important equation, and to state its precise scope and limitations.

We shall see that the equation is not true, in general, when the point Q is chosen at random, and that it is not even true, in general, when Q is the instantaneous center; on the other hand, we shall see that it is true for certain other choices of the point Q , which are not usually mentioned.

For the sake of clearness, we shall distinguish two distinct theorems, the first of which may be called the theorem of moments about a point fixed in the body, and the second, the theorem of moments about a point fixed in the plane. The proof of these theorems, which is ordinarily made to depend on the general theory of angular momentum (couched in a series of rather long and forbidding equations), is here developed from much more elementary considerations, and by steps which, it is believed, can be readily recalled to mind by any student who has once been through the reasoning.

Theorem I. *Theorem of moments about a point fixed in the body.* If a rigid body W is free to move in a plane, under the action of any external forces, and if Q is any *suitably chosen* point of the body (see below), then, at any instant,

¹ Of the recent elementary text books which have come under my notice, the following widely used books are among those which take the extreme view that the equation of rotation is always true, for any point Q : SMITH and LONGLEY, *Theoretical Mechanics*, p. 236; H. M. DADOURIAN, *Analytical Mechanics*, 1913, § 184 and § 188; FULLER and JOHNSTON, *Applied Mechanics*, vol. 1, 1913, pp. 308–309. The statement of Fuller and Johnston is particularly explicit on this point: “When a body has imparted to it a plane motion of rotation about a *moving or a fixed axis*, the motion may be resolved into a motion of rotation relative to an axis perpendicular to the plane of motion through *any point A*, with an angular velocity ω equal to that of the body about the axis of rotation, and a motion of translation with a linear velocity V_a equal to that of the point A The resultant of the system of forces required to impart to a rigid body the motion described above may be determined by finding the resultant of the system of forces required to impart the motion of rotation relative to the axis A , and the resultant of the system of forces required to impart the motion of translation, and combining the two resultants. *While the point A may be any point within the rotating body*, in most problems dealing with motion of this kind the simplest solution can be made by taking A at the center of gravity.” (Italics are mine.) This is a particularly good example of the confusion which may result from speaking of the angular velocity of a body *about an axis*, when the axis is not fixed. The angular velocity of a body should be defined without reference to any axis of rotation, as simply the rate of change of the angle which a line fixed in the body makes with a line fixed in the plane.

$$\Sigma Fp = \frac{W}{g} k^2 \frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the point Q (in the positive direction of rotation) at the instant in question, and $d\omega/dt$ = the angular acceleration of the body at that instant (θ being measured in radians, in the positive direction of rotation, and $\omega = d\theta/dt$); while k = the radius of gyration of the body about the point Q (that is, about an axis through Q perpendicular to the plane of the motion).

Here by a "suitably chosen" point Q we mean any point Q of the body such that (1) Q is fixed in the plane; or (2) Q is moving with uniform velocity in a straight line; or (3) Q is the center of mass; or (4) Q has an acceleration whose direction, at the instant in question, passes through the center of mass; and the equation above will be valid when and only when Q has some one of these four properties.

Proof of Case 1. Entirely elementary proofs for the case of rotation about a fixed axis are readily accessible and need not be repeated here. We shall mention only that this case can be conveniently recalled to mind when stated in the following form: If a rigid body is free to rotate about a fixed axis, under the action of any turning moment, the angular acceleration at any instant is the same as if all the material of the body were concentrated in a single particle at a distance k from the axis, where k is the radius of gyration of the body about the axis.

Proof of Case 2. First, regard the given body as a system of free particles acted on by definite external and internal forces, under given initial conditions (the internal forces being just sufficient to keep the system rigid); and let V_x , V_y be the initial values of the velocity components of the given point Q , referred to fixed axes in the plane.

Now imagine the initial conditions so changed that the initial velocity components of each and every particle are diminished by the amounts V_x and V_y , all the forces remaining the same as before. In the new situation thus imagined, it is easy to see: (1) that the system will still move as a rigid body, and (2) that its angular velocity and angular acceleration will remain unchanged; but the point Q will be reduced to rest.

Under these new conditions, therefore, we may apply the theorem of case 1, thus obtaining the equation $\Sigma Fp = (W/g)k^2 d\omega/dt$, where all the letters have their original meanings. The truth of the equation is thus proved for case 2.

Proof of Case 4, which includes case 3. Regard the given body again as a system of free particles, as in case 2, and let a_x and a_y be the acceleration components of the given point Q , referred to fixed axes in the plane.

Now imagine impressed upon each and every particle w an additional force with components $-(w/g)a_x$ and $-(w/g)a_y$; this will clearly have the effect of diminishing the acceleration components of each particle by the amounts a_x and a_y . In the new situation thus imagined, it is easy to see: (1) that the system will still move as a rigid body, and (2) that its angular velocity and angular

acceleration will remain unchanged; but the point Q will be reduced to a state of uniform motion in a straight line.

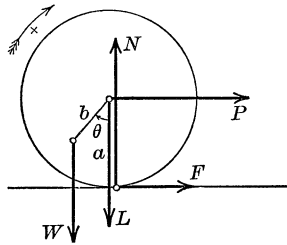
Under these new conditions, therefore, we may apply the theorem of case 2, and thus obtain the equation

$$\Sigma Fp + \Sigma f'p' = (W/g)k^2 d\omega/dt,$$

where ΣFp and ω have their original meanings, and $\Sigma f'p' =$ the sum of the moments about Q of the additional impressed forces.

Now it will be noticed that these additional forces form, at each instant, a set of parallel forces each of which is proportional to the mass of the particle on which it acts. Hence, by the definition of the center of mass, in finding the sum of the moments of these parallel forces, we may replace the entire set of forces by a single force F' acting at the center of mass. Furthermore, the direction of these parallel forces, and hence the direction of F' , is opposite to the direction of the vector acceleration of the point Q ; but, by the hypothesis of case 4, this direction lies along a line drawn through Q and the center of mass. Hence the moment of F' about Q will be zero; that is, $\Sigma f'p' = 0$, and the theorem is established.

It should be carefully noted that in the statement of theorem I the instantaneous center—that is, the point of no velocity—is not included in the list of “suitable” points Q ; in other words, it is not legitimate to “take moments about the instantaneous center,” unless this point happens to have also one of the other properties explicitly mentioned in the theorem.



To illustrate the danger of “taking moments about the instantaneous center,” consider the following simple problem.

Suppose a carriage is moving with uniform velocity along a level road, and consider the motion of a wheel, of radius a , in which the center of mass does not coincide with the center of the wheel. The center of mass will then move in a trochoidal path, and the components of its acceleration will be $\ddot{x} = b\omega^2 \sin \theta$, $\ddot{y} = b\omega^2 \cos \theta$, where b is the distance from the hub to the center of mass, and ω the constant angular velocity of the wheel.

The forces acting on the wheel are (1) the weight W , acting downward at the center of mass; (2) the pressure of the bearings, which we may suppose resolved

into a forward component P , and a downward component L ; and (3) the reaction of the road, which we may suppose resolved into a forward component F , and an upward component N . The equations of motion for the center of mass are therefore

$$P + F = (W/g)b\omega^2 \sin \theta, \quad (1)$$

$$N - W - L = (W/g)b\omega^2 \cos \theta. \quad (2)$$

Since the angular velocity is constant, $d\omega/dt = 0$, and the equation of rotation may be written as follows:

(a) about the center of mass:

$$Pb \cos \theta + (L - N)b \sin \theta - F(a - b \cos \theta) = 0; \quad (3a)$$

(b) about the hub, which is moving with uniform velocity in a straight line:

$$Fa + Wb \sin \theta = 0; \quad (3b)$$

(c) about the lowest point of the wheel, which is the instantaneous center:

$$Pa - Wb \sin \theta = 0. \quad (3c)$$

Of the last three equations, (3a) and (3b) are true, in accordance with theorem I, and from either of them, combined with (1) and (2), we find

$$Pa - Wb \sin \theta = \frac{W}{g} ab \omega^2 \sin \theta. \quad (4)$$

But this shows that equation (3c) is false. Hence in this case, wholly erroneous results would be obtained by taking moments about the instantaneous center.

If the center of mass had coincided with the hub, it would then indeed have been legitimate to take moments about the lowest point of the wheel, not, however, because this point is the instantaneous center, but because it is in this case a point whose vector acceleration at the instant in question passes through the center of mass (case 4). The use of case 4 and case 2 will often lead to simpler equations that could be obtained if we confined ourselves to the use of the more familiar cases 1 and 3.

Corollary I. In discussing the motion of a rigid body in a plane, any given set of external forces may be replaced by (1) a single force, acting at the center of mass, and (2) a couple, acting anywhere in the body. The components of the single force will be $(W/g)a_x$ and $(W/g)a_y$, where a_x and a_y are the components of the acceleration of the center of mass. The moment of the couple will be $(W/g)k^2 d\omega/dt$, where $d\omega/dt$ is the angular acceleration of the body, and k its radius of gyration about the center of mass.

The single force, acting at the center of mass, will have no effect on the angular velocity of the body; while the couple will have no effect on the motion of the center of mass.

It is clear, however, from theorem I, that a single force whose line of action does not pass through the center of mass will produce not only a change in the motion of the center of mass, but also a change in the angular velocity of the body.

Corollary II. In discussing the motion of a rigid body in a plane, the point of application of any force may be shifted at pleasure along the line of action of that force.

For the moment of the force about any point, and the components of the force along any axes will be unchanged.

This corollary, known as the principle of transmissibility of forces, is usually assumed as a fundamental principle of statics. But this assumption is not required for the proof of the theorems here stated.

Theorem II. *Theorem of moments about a point fixed in the plane.* If a rigid body W is free to move in a plane, under the action of any external forces, and if O is any point fixed in the plane, then, at any instant,

$$\Sigma Fp = \frac{W}{g} \frac{d(h\bar{v})}{dt} + \frac{W}{g} \bar{k}^2 \frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the point O (in the positive direction of rotation) at the instant in question, and

$h\bar{v}$ = the moment about O of the velocity $\bar{v}$ of the center of mass at that instant; while

$\bar{k}$ = the radius of gyration of the body about its center of mass.

Proof. Let C be the point of the plane through which the center of mass is passing at the instant in question, and let a_r and a_θ be the acceleration components of the center of mass along and perpendicular to the fixed axis OC at that instant.

By corollary I, the external forces acting on the body may be replaced by (1) a single force, acting at the center of mass, with components $(W/g)a_r$ and $(W/g)a_\theta$ along and perpendicular to OC ; and (2) a couple, acting anywhere in the body, with a moment (about any point) equal to $(W/g)\bar{k}^2 d\omega/dt$. Hence the sum of the moments of the external forces about O reduces to

$$\Sigma Fp = (W/g)ra_\theta + (W/g)\bar{k}^2 d\omega/dt,$$

where $r = \overline{OC}$. But from the elementary kinematics of a particle in polar coordinates, $a_\theta = \frac{1}{r} \frac{d(r\bar{v}_\theta)}{dt}$, where $\bar{v}_\theta = \frac{r d\theta}{dt}$ = the velocity component of the center

of mass in the direction perpendicular to OC . Also, $r\bar{v}_\theta = h\bar{v}$, where h is the perpendicular from O to the vector $\bar{v}$. Hence

$$\Sigma Fp = (W/g)d(h\bar{v})/dt + (W/g)\bar{k}^2 d\omega/dt,$$

which establishes the theorem.

This theorem II is by no means so simple or so important as theorem I. It is useful chiefly in problems in impact, especially in the derived form obtained by multiplying through by dt and integrating from $t = t_0$ to $t = t_1$. In this derived form the equation is known as the equation of impulse and momentum. But neither of these concepts is required for the proof as here given.